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Strong coupling phenomena on the noncommutative plane

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Abstract: We study strong coupling phenomena in U(1) gauge theory on the non-commutative plane. To do so, we make use of a T-dual description in terms of an $N \to \infty$ limit of U(N) gauge theory on a commutative torus. The magnetic flux on this torus is taken to be $m = N - 1$, while the area scales like $1/N$, keeping Λ_{QCD} fixed. With a few assumptions, we argue that the speed of high frequency light in pure non-commutative QED is modified in the non-commutative directions by the factor $\Lambda^4_{\mathrm{QCD}}\theta^2$, provided $\Lambda^2_{\mathrm{QCD}}\theta \gg 1$. If charged flavors are included, there is an upper bound on the momentum of a photon propagating in the non-commutative directions, beyond which it is unstable against production of charged pairs. We also discuss a particular $\theta \to \infty$ limit of pure non-commutative QED which is T-dual to a more conventional $N \to \infty$ limit with m/N fixed. In the non-commutative description, this limit gives rise to an exotic theory of open strings.

Keywords: 1/N Expansion, Duality in Gauge Field Theories, Brane Dynamics in Gauge Theories, Non-Commutative Geometry.

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1. Introduction

In this paper, we study strong coupling aspects of U(1) gauge theory on the non-commutative plane. We emphasize that our intent is not to examine possible experimental signatures of non-commutativity. It is known [1, 2] that a non-commutative version of QED is inconsistent with experiment unless there is a cutoff which is small compared to the non-commutativity scale $1/\sqrt{\theta}$. In the absence of a cutoff, there are large lorentz violating operators generated at one loop, even for arbitrarily small $|\theta|$, due to UV/IR mixing. It is also impossible to reproduce the observed infrared stability of QED. The phenomena which we shall study arise in the absence of a cutoff, and are therefore primarily of theoretical interest. In this case the pure non-commutative U(1) gauge theory is asymptotically free [3] and the effective coupling grows at large distance, leading to interesting strong coupling effects.

One tool which we will use to study such effects is T-duality, or Morita equivalence. On a non-commutative two-torus, which we take to be square for simplicity, there is an $\mathrm{SL}(2,\mathbb{Z})$ duality group relating $\mathrm{U}(N)$ theories of different rank N, magnetic flux m, torus radius R, and non-commutativity parameter θ. This duality is inherited from T-duality in string theory [4, 5], however it can be understood entirely in the context of Yang-Mills theory [6]–[11].

We shall begin by considering pure QED (no charged matter) on $\mathcal{R}^4$ with two non-commuting spatial directions; $[x^1, x^2] = i\theta^{12} \equiv \theta$. The gauge invariant photon creation operators in this theory are open Wilson lines [12]–[15], with separation between the end-points given by $L^i = \theta^{ij} p_j$, where p is the momentum. The dispersion relation for photons at large momentum depends crucially on whether or not these open Wilson lines have zero tension. Pure non-commutative QED (NC-QED) shares much in common with conventional QCD, such as gauge field self interactions and asymptotic freedom [3]. We shall conjecture that this similarity also includes a non-vanishing string tension in a strong coupling region $\Lambda^2\theta \gg 1$. The relation between NC-QED and QCD can be understood precisely by making use of T-duality. However, unlike asymptotic freedom, the existence of a non-vanishing string tension in NC-QED is not manifest from this duality, since the dual description is a zero area limit of QCD on a torus with fixed $\Lambda_{\rm QCD}$. Naively one would expect the string tension to vanish and the electric flux to condense below a critical radius.

To make use of T-duality, we shall view QED on the non-commutative plane as the infinite area limit of QED on $T^2_\theta \times \mathcal{R}^2$, where T^2_θ is a non-commutative two-torus. This limit is T-dual to an $N \to \infty$ limit of U(N) gauge theory on a commutative $T^2 \times \mathcal{R}^2$. There is a non-zero magnetic flux on the dual T^2, whose radius R' scales like $1/\sqrt{N}$ with $\Lambda_{\rm QCD}$ fixed. In the absence of magnetic flux, such a limit would lead to a truncation to the zero modes on the torus, i.e. dimensional reduction. However because of the non-zero flux, there are twisted boundary conditions giving rise to fractional momentum modes which do not decouple in the large-N–small-area limit. As will be clear later, the U(1) part of the dual U(N) theory plays a trivial role, describing the physics of a free decoupled sector of pure NC-QED. The interesting dynamics resides in the SU(N) component of the dual theory.

While pure QED on the non-commutative plane can be obtained as an $N \to \infty$ limit of a conventional gauge theory, this is not in general a planar limit. It differs from the usual 't Hooft large-N limit because of the non-zero magnetic field and the $1/N$ scaling of the area of the torus area. In this case, finite size effects ruin the usual large-N counting.

In the $N \to \infty$ limit under consideration, the torus becomes small compared to $1/\Lambda_{\rm QCD}$. Nevertheless, one still encounters strong coupling phenomena. Strong coupling physics enters, for instance, into the computation of the energy of a sufficiently large electric flux on the torus. This is somewhat counter-intuitive. If N were fixed, the energy of any electric flux on the torus could be computed perturbatively for a sufficiently small radius. Perturbatively, this energy is zero when the third (commutative) spatial direction is non-compact. One might also naively expect electric flux condensation on a sufficiently small torus, via spontaneous breaking of the $Z_N \times Z_N$ group of large gauge transformations.

The loophole is that, although we take a zero radius limit, we also simultaneously take $N \to \infty$, in which case one can not necessarily compute the electric flux energy perturbatively. At fixed N there is a limit to how large the electric flux can be, being defined only modulo N. In the large-N–small-radius limit, we may consider increasingly large electric fluxes whose energies are not perturbatively computable. Furthermore, the usual argument for Z_N symmetry breaking at small radius does not apply, due in part to the presence of a non-zero magnetic flux. We conjecture that in the strongly coupled region

$\Lambda^2\theta \gg 1$ the $Z_N \times Z_N$ symmetry is unbroken and the tension of QCD strings wrapping the torus is non-vanishing. In NC-QED, this implies that open Wilson lines have a non-zero tension. The photons of NC-QED, which are created by such open Wilson lines, are dual to QCD strings wrapping the torus $\sim \sqrt{N}$ times. Such strings have fixed length and energy in the $N \to \infty$ limit with $R' \sim 1/\sqrt{N}$.

Since the length of open Wilson lines is proportional to their transverse momentum, a non-zero string tension leads to a dramatically modified dispersion relation for photons with large momentum in the non-commutative plane. At short wavelengths, we shall find that the speed of light in the non-commutative plane is modified by the factor $\Lambda^4\theta^2$, where the dynamically generated scale Λ is proportional to $\Lambda_{\rm QCD}$ in the dual description. A similar phenomenon occurs at the *classical* level in non-commutative QED, in which case the speed of light depends on the direction when a background magnetic field is turned on [16, 17].

When charged matter is included in NC-QED, there is an upper bound on the momentum of a photon propagating in a non-commutative direction. At sufficiently large momentum, at which the propagation speed is "superluminal", the photon is unstable and decays by e_+e_- pair production. This decay is analogous to the instability of high energy photons in a strong magnetic field in conventional QED, although the mechanism is quite different. For sufficiently large momentum, the open Wilson line associated with a photon becomes long enough that it is energetically favorable to break off segments of Wilson line bounded by electron-positron pairs, the length of which is not constrained to be proportional to the transverse momentum. A similar phenomenon would occur in NC-QCD. Glueballs in NC-QCD are also open Wilson lines with length proportional to the momentum in the non-commutative plane. Above a critical momentum, they have an added instability against decay to mesons.

In [3] it was argued that stringy behavior of a non-commutative field theory arises as $\theta \to \infty$, since non-planar graphs vanish in this limit. For a *gauge* theory, the physical content of such a theory is not obvious. In the pure non-commutative gauge theory, there are no local gauge invariant operators. Furthermore the open Wilson lines, which are quasi-local operators, have endpoint separations satisfying $L^i = \theta^{ij}p_j$, and are therefore ill defined in the $\theta \to \infty$ limit. Since only zero momentum operators survive as $\theta \to \infty$, space-time is in some sense absent from the string theory which arises in this limit.

There is however a modified large-θ limit in which there exist well behaved gauge invariant operators with non-zero momentum. On a non-commutative torus, open Wilson lines are characterized not just by their momentum, but also by a "winding number" or electric flux. In the limit $R \to \infty$, where R is the radius of the torus, electric flux contributes an infinite amount to the length of the open Wilson line, just as momentum does in the infinite θ limit. The sum of these two infinite contributions may be finite provided θ and the area are sent to infinity simultaneously. In such a limit, one obtains an exotic open string theory. The string coupling of this theory is given by l_s^2/θ, where the string scale $l_s = \theta/R$ and is held fixed in the large-θ limit.

This open string theory is T-dual to a conventional large-N limit of U(N) gauge theory on a torus of fixed size and fixed non-zero m/N, where m is the magnetic flux. In some instances, such as pure two dimensional NC-QED or strongly coupled four dimensional $\mathcal{N}=4$ NC-QED, it is possible to compute correlation functions of open strings exactly to leading order in the string coupling by using the dual $\mathrm{U}(\infty)$ description.

The organization of this paper is as follows. In section 2 we give a brief review of gauge theory on a non-commutative space and the T-duality symmetries of gauge theory on a non-commutative torus. In section 3 we construct the $\mathrm{U}(\infty)$ dual of pure QED on the non-commutative plane. In section 4 we use this dual description to suggest that open Wilson lines have non-zero tension at strong coupling, and compute the photon dispersion relation. We then argue there is an upper bound on the photon momentum when charged matter is included. Section 5 contains a discussion of the difference between the $N\to\infty$ limit dual to NC-QED, and the usual 't Hooft $N\to\infty$ limit. In section 6 we construct an open string theory from a $\theta\to\infty$ limit of NC-QED. We then present some exact open string correlators in the free string limit.

2. Review of non-commutative gauge theories and Morita equivalence

2.1 Non-commutative gauge theory

Field theory on a noncommutative space can be constructed by replacing products of functions with Moyal $*$-products. The $*$-product is defined as follows;

$$f(x)*g(x)=e^{i\frac{\theta^{\mu\nu}}{2}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}}f(x)g(y)|_{y\to x}\,. \tag{2.1}$$

The action of the pure noncommutative $\mathrm{U}(p)$ Yang-Mills theory is

$$S=\frac{1}{g^2}\int d^Dx\,\mathrm{Tr}(F_{\mu\nu}(x)*F^{\mu\nu}(x))\,, \tag{2.2}$$

where

$$F_{\mu\nu}(x)=\partial_\mu A_\nu-\partial_\nu A_\mu-i(A_\mu*A_\nu-A_\nu*A_\mu)\,. \tag{2.3}$$

In much of what follows, we shall take $\theta_{12}=-\theta_{21}=\theta$ to be non-zero with all other components vanishing. When the 1-2 plane is compactified on a torus of radius R, the non-commutative parameter becomes a periodic variable $\theta\sim\theta+4\pi R^2$, and it will be convenient to define the dimensionless quantity $\Theta\equiv\theta/(4\pi R^2)$.

The pure non-commutative gauge theory has no local gauge invariant operators. Instead, there are quasi-local gauge invariant operators constructed from open Wilson lines with possible insertions of local gauge covariant operators [12]–[15]. The open Wilson lines are gauge invariant provided the separation between the endpoints is $L^i=\theta^{ij}p_j$, where p is the momentum.

As an example, a straight open Wilson line in the 2 direction of the non-commutative 1-2 plane is given by

$$\int dx^1dx^2P^*\left(e^{i\int_C A_2}\right)*e^{ip_1x^1}\,, \tag{2.4}$$

where C is a path between the points (x^1,x^2,x^3) and $(x^1,x^2+\theta^{12}p_1,x^3)$, and P^* indicates that the exponential is path ordered with respect to the non-commutative $*$-product. An example of a gauge invariant operator with a non-trivial $\theta \to 0$ limit is

$$\int d^3x P^* \left(F_{\mu\nu} e^{i \int_C A_2} \right) * e^{i\mathbf{p}\cdot\mathbf{x}} , \tag{2.5}$$

where the local operator $F_{\mu\nu}$ is inserted anywhere within the open Wilson line.

On a torus the open Wilson line may also have an integer winding number (electric flux) e_i [11]. Gauge invariance again requires the endpoints to be separated by $L^i = \theta^{ij} p_j$, however the minimum length with an electric flux e_i is $\sqrt{\mathcal{L}_i \mathcal{L}_i}$ where $\mathcal{L}_i = \theta^{ij} p_j + 2\pi R e^i$ and R is the radius of the (square) torus. Non-commutative electric fluxes will play a role in our discussion of the $\theta \to \infty$ limit.

2.2 T-duality

$\mathrm{U}(N)$ gauge theory on a non-commutative T^2 exhibits an $\mathrm{SL}(2,\mathbb{Z})$ Morita equivalence which is inherited from string theory T-duality (see [4, 5]), but which can be demonstrated explicitly without recourse to string theory [6]–[11]. This is not a duality in the traditional sense, as it exists at the classical level. The duality acts as follows:

$$\begin{aligned}
\begin{pmatrix} N \\ m \end{pmatrix} &\to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} N \\ m \end{pmatrix} && (2.6)\\
\Theta &\to \frac{c + d\Theta}{a + b\Theta} && (2.7)\\
R^2 &\to R^2 (a + b\Theta)^2 && (2.8)\\
g^2 &\to g^2 (a + b\Theta) && (2.9)\\
\Phi &\to (a + b\Theta)^2 \Phi - b(a + b\Theta) && (2.10)
\end{aligned}$$

where $\Theta \equiv \theta/(4\pi R^2)$, m is the magnetic flux, g is the gauge coupling, and R is the radius of the torus. The action is

$$S = \frac{1}{g^2} \int d^D x \, \mathrm{tr}(F_{\mu\nu} - 2\pi \Phi_{\mu\nu} I)^2 , \tag{2.11}$$

where $\Phi_{\mu\nu}$ is a constant antisymmetric tensor background for which all components vanish except those with indices on the non-commutative torus $\Phi_{12} = \Phi_{21} \equiv \Phi$.

3. The non-commutative plane from an $N \to \infty$ limit

We will study U(1) gauge theory on the non-commutative plane by taking an infinite volume limit of the theory on a non-commutative torus of radius R. The first Chern class and background term Φ shall be set to zero. We consider a rational non-commutative parameter, $\Theta = -c/N$, where c and N are integers with greatest common divisor one. When taking the infinite volume limit, $\theta = 4\pi R^2 \Theta$ will be held fixed. There is no loss of generality in considering rational Θ, since one can still obtain any $\theta = 2\pi R^2 \Theta$ in the infinite R limit.

For $\Theta = -c/N$, there is a dual description in terms of a $\mathrm{U}(N)$ gauge theory on a commutative torus with non-zero magnetic flux. The magnetic flux m is a solution of the equation $aN - cm = 1$ for integer a. The relations (2.6) - (2.10) then become

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} a & m \\ c & N \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} N \\ m \end{pmatrix} \tag{3.1}$$

$$\Theta = -\frac{c}{N} \to \Theta' = \frac{c + N\Theta}{a + m\Theta} = 0 \tag{3.2}$$

$$R^2 \to R'^2 = R^2 \left(a - m\frac{c}{N}\right)^2 = \frac{R^2}{N^2} \tag{3.3}$$

$$g^2 \to g'^2 = g^2 \left(a - m\frac{c}{N}\right) = \frac{g^2}{N} \tag{3.4}$$

$$\Phi = 0 \to \Phi' = -m\left(a - m\frac{c}{N}\right) = -\frac{m}{N} \tag{3.5}$$

The equation $aN - cm = 1$ only fixes m modulo N. For a $\mathrm{U}(N)$ bundle, the first Chern class m is equal, modulo N, to the 't Hooft magnetic flux in the $\mathrm{SU}(N)/Z_N$ sector of the theory, which is defined only modulo N (see for instance [19]). In the $\mathrm{U}(N)$ description, the background term Φ' is equal to the first Chern class, so that $\int_{T^2} \mathrm{tr}(F' - \Phi' I) = 0$. There is a symmetry which simultaneously shifts m by N and the Φ by 1. This symmetry relates all the solutions of $aN - cm = 1$ for a given c and N. Note that the action is the same as that of a $\mathrm{U}(1) \times \mathrm{SU}(N)/Z_N$ gauge theory in a sector with vanishing magnetic flux in the $\mathrm{U}(1)$ component and 't Hooft flux m in the $\mathrm{SU}(N)/Z_N$ component. The dynamics in the $\mathrm{U}(1)$ component is trivial. This sector is dual to a free decoupled sector of pure NC-QED containing photons with momenta Nk_i/R in the non-commutative plane, where k_i are integers. The interesting dynamics resides is in the $\mathrm{SU}(N)/Z_N$ component.

In order to take the $R \to \infty$ limit with $\theta = \Theta 4\pi R^2$ fixed, one must take $R'N \to \infty$ keeping $cR'^2 N$ fixed. One way to do this is to take the $N \to \infty$ limit with $c = 1$, $m = N - 1$ and $4\pi R'^2 = \theta/N$. Note that $g'^2 N$ is equal to the NC-QED coupling g^2 and should therefore be held fixed in the large-N limit. If not for the $1/\sqrt{N}$ scaling of the radius R', this would be a conventional 't Hooft large-N limit. Λ_{QCD} will be held fixed as $N \to \infty$. This is necessary so that the dynamically generated scale Λ in the non-commutative $\mathrm{U}(1)$ description is also held fixed in the large-R limit. At the quantum level, the relation $g^2 = g'^2 N$ makes sense if the couplings are evaluated at the same scale. To one loop, the running coupling of the non-commutative $\mathrm{U}(1)$ theory is [3]

$$\frac{1}{g^2(\mu)} = \frac{1}{16\pi^2}\frac{11}{3}\ln\frac{\mu}{\Lambda}\,, \tag{3.6}$$

which is valid at scales μ above the non-commutativity scale

$$\mu > \frac{1}{\sqrt{\theta}} = \frac{1}{\sqrt{4\pi N}R'}\,. \tag{3.7}$$

For scales $\mu > 1/R' = N/R$, the dual $\mathrm{SU}(N)$ description is more appropriate (see [21]), in which case the one loop running of the 't Hooft coupling $\lambda = g'^2 N$ is given by

$$\frac{1}{\lambda(\mu)} = \frac{1}{16\pi^2}\frac{11}{3}\ln\frac{\mu}{\Lambda_{\mathrm{QCD}}}\,. \tag{3.8}$$

The running couplings in these two regions are smoothly matched if $\Lambda = \Lambda_{\rm QCD}$.

The non-commutative gauge coupling is asyptotically free. In the case $\Lambda^2\theta \ll 1$, the running stops below the non-commutativity scale $1/\sqrt{\theta}$. However in the region $\Lambda^2\theta \gg 1$, the coupling becomes large in the infra red before reaching the non-commutativity scale. We emphasize that the significance of the running coupling in the region $\Lambda^2\theta \gg 1$ is not clear due to UV/IR mixing phenomena. The basic objects of the theory are the open Wilson lines, whose length is proportional to their momentum, $L = p\theta$. Thus for $\Lambda^2\theta \gg 1$, one can never set all relevent physical scales (inverse lengths and momenta) to be greater than Λ. Only for $\Lambda^2\theta \ll 1$ is there truly a weakly coupled sector of the theory.

Although the radius in the U(N) dual of NC-QED is scaled to zero in the large-N limit, this scaling does not amount to a dimensional reduction. The reason is that the gauge fields in the U(N) description A' satisfy twisted boundary conditions which permit fractional momenta. It will be convenient to define the field

$$\tilde{A}'_\mu \equiv A'_\mu - \delta_{\mu 1}\frac{mx^2}{N2\pi R'^2} I\,. \tag{3.9}$$

We will choose boundary conditions such that $\mathrm{tr}\,\tilde{A}'_\mu$ is periodic, and

$$\tilde{A}'(x^1+2\pi R, x^2) = P^{-m}\tilde{A}'(x^1,x^2)P^m \tag{3.10}$$

$$\tilde{A}'(x^1, x^2+2\pi R) = Q\tilde{A}'(x^1,x^2)Q^{-1} \tag{3.11}$$

where

$$Q = \begin{pmatrix} 1 & & & \\ & e^{\frac{2\pi i}{N}} & & \\ & & \ddots & \\ & & & e^{\frac{2\pi i(N-1)}{N}} \end{pmatrix} \tag{3.12}$$

and

$$P = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & & \ddots \\ 1 & & & \end{pmatrix} \tag{3.13}$$

Q and P satisfy

$$PQ = QPe^{2\pi i/N}\,. \tag{3.14}$$

These boundary conditions correspond to a sector with 't Hooft magnetic flux (and first Chern class) m. Since $(P^m)^N = Q^N = 1$, the gauge field $\tilde{A}'$ is strictly periodic on an N^2 cover of the torus, with periods $2\pi NR$. Because the greatest common divisor of N and m is 1 in our case, there is no smaller multiple cover on which the gauge fields are periodic. Note that the U(1) ($\sim$ identity) component of $\tilde{A}'$ is periodic on the torus so that, in the large-N limit which we consider, only the zero modes of the U(1) sector survive. In the SU(N)/Z_N sector however, the gauge field can not be written as a sum of periodic fluctuations and a non-periodic background. There are modes of finite (fractional) momenta which do not decouple in the large-N/small radius limit. For completeness, the

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explicit map [9, 10, 11] between the gauge field A of non-commutative QED gauge and the U(N) gauge field A' is as follows. A general solution of the boundary conditions (3.11) is

$$\tilde{A}' = \sum_{\vec{r}} a^{\mu}_{\vec{r}} Q^{-cr_1} P^{r_2} \exp\left(-i\pi \frac{c}{N} r_1 r_2\right) \exp\left(-i\frac{\vec{r}\cdot\vec{x}}{NR'}\right). \tag{3.15}$$

where the independent variables $a^{\mu}_{\vec{r}}$ are the Fourier modes of the non-commutative gauge field,

$$A = \sum_{\vec{r}} a^{\mu}_{\vec{r}} \exp\left(-i\frac{\vec{r}\cdot\vec{x}}{R}\right). \tag{3.16}$$

Despite the twisted boundary conditions, the gauge invariant operators of the U(N) theory obey periodic boundary conditions. These operators can in general be split into U(1) and SU(N) parts. These parts are separately periodic for gauge invariant local operators and Wilson loops with trivial homology. With the exception of modes with zero momentum on the torus, such operators create states with infinite energy in the $N \to \infty$ limit with $R' \sim 1/\sqrt{N}$, and may be integrated out. On the other hand, the Wilson loops with non-trivial homology factorize into U(1) and SU(N)/Z_N parts which have fractional momenta. These operators will be of particular interest, as they correspond to open Wilson lines in the non-commutative description [10, 11, 9]. Due to the fractional momenta of the separate components, these operators do not decouple in the large-N–small-radius limit.

We will now make the above discussion more precise. In the U(1) sector of the U(N) theory, the gauge field can be written as a sum of a non-periodic background term (accounting for the non-zero first Chern class) and a periodic piece which is integrated out, with the exception of the zero modes on the torus. Consider a Wilson loop in the fundamental representation of U(N) which wraps the torus e_2 times in the 2 direction. The holonomy is given by

$$W_{e_2}(x^1) = \mathrm{tr}\left[P e^{i\oint A'_2} \Omega_2^{e_2}\right], \tag{3.17}$$

where Ω_2 is a U(N) transition function associated with the twisted boundary conditions in the 2 direction, and is required for gauge invariance [20]. For the boundary conditions we have chosen,

$$\Omega_2^{e_2} = P^{-e_2} e^{-i\frac{m e_2 x^1}{NR}}. \tag{3.18}$$

The operator (3.17) is periodic in x^1, and one can define Wilson loops with transverse momentum n_1/R';

$$\mathcal{W}_{e_2,n_1} \equiv \int_0^{2\pi R'} dx^1 W_{e_2}(x^1) e^{i\frac{n_1 x^1}{R'}}, \tag{3.19}$$

where n_1 is an integer. Splitting this into SU(N) and U(1) components gives

$$\mathcal{W}_{e_2,n_1} = e^{iC_2(x^0,x^3)} \int_0^{2\pi R'} dx^1 e^{i\frac{(n_1+\frac{m}{N}e_2)x^1}{R'}} \tilde{\mathcal{W}}_{e_2}(x^1), \tag{3.20}$$

where $\tilde{W}$ is the SU(N) component of the holonomy, involving SU(N) valued transition functions, and $C_\mu(x^0,x^3)$ is the zero mode of the U(1) component of the gauge field;

$$C_\mu(x^0,x^3) = \frac{1}{N}\int_{T^2} \mathrm{tr}\, \tilde{A}'_\mu. \tag{3.21}$$

The U(1) field $\mathrm{tr}\,\tilde{A}'$ is periodic, so except for the zero modes, the U(1) sector decouples in the large-N limit with $R' \sim 1/\sqrt{N}$. On the other hand, the SU(N) component of the holonomy $\tilde{W}(x^1)$ is periodic only over an N-fold cover of the torus,

$$\tilde{W}_{e_2}(x^1 + 2\pi R') = \tilde{W}_{e_2}(x^1) e^{2\pi i \frac{m}{N} e_2} \,. \tag{3.22}$$

We can therefore define (fractional) Fourier modes

$$\tilde{\mathcal{W}}_{e_2,k_1} \equiv \frac{1}{N} \int_0^{2\pi R' N} dx^1 \tilde{W}(x^1) e^{i\frac{k_1 x^1}{R'}} \tag{3.23}$$

with k_1 an integer multiple of $1/N$. Then

$$\mathcal{W}_{n_1,e_2} = e^{-iC_2} \tilde{\mathcal{W}}_{e_2, n_1 - \frac{m}{N} e_2} \,. \tag{3.24}$$

The fractional momentum $(n'_1 - \frac{m}{N} e'_2)/R'$ may be finite in the large-N limit with $R' \sim 1/\sqrt{N}$, so the states associated with Wilson loops of non-trivial homology can not necessarily be integrated out. For $m = N-1$, one can get finite momentum by choosing $n_1 = e_2 \sim \sqrt{N}$.

4. The non-commutative photon

4.1 The dual of the non-commutative photon

We now consider photon propagation in three spatial dimensions, with non-commutativity in the 1-2 plane. On the commutative plane, the photon is created by a local gauge invariant operator such as $F_{\mu\nu}$. On the non-commutative plane the natural generalization is the open Wilson line with an insertion of $F_{\mu\nu}$, as in (2.5).

The explicit map between open Wilson lines in non-commutative QED at rational Θ and operators of the dual commutative U(N) theory is discussed in [11, 10, 9], and we review it briefly below. For simplicity, we restrict ourselves to open Wilson lines in the 2 direction, which are dual to U(N) Wilson loops in the fundamental representation wrapping in the 2 direction. The open Wilson lines are characterized by integers (n_1, e_2) labelling the transverse momentum and winding (or electric flux) respectively. Similarly the holonomies of the commutative theory are characterized by a transverse momentum and winding number (n'_1, e'_2). Note that n'_1 is the momentum of the full U(N) holonomy, which is integer rather than fractional. The map between the non-commutative open Wilson lines and the commutative holonomies is given by

$$\begin{pmatrix} e_2 \\ n_1 \end{pmatrix} = \begin{pmatrix} a & -c \\ -m & N \end{pmatrix} \begin{pmatrix} e'_2 \\ n'_1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1-N & N \end{pmatrix} \begin{pmatrix} e'_2 \\ n'_1 \end{pmatrix} . \tag{4.1}$$

This map is valid with or without local operator insertions.

Since we are considering the large volume limit of the non-commutative theory, we will take $e_2 = 0$. Via (4.1), this implies $e'_2 = n'_1 = n_1$. We wish to keep the momentum $p_1 = n_1/R$ fixed in the large-N limit with $R \sim \sqrt{N}$, therefore $n_1 = n'_1 = e'_2 \sim \sqrt{N}$. The wrapped Wilson loop in the U(N) description has length $2\pi R' e'_2$ which is fixed in

NC-QED on $T^2_\theta \times \mathcal{R}^2$	U(N) YM on $T^2 \times \mathcal{R}^2$
magnetic flux 0	$m = N - 1$
radius $R = \sqrt{\theta N}$, θ fixed	$R' = \sqrt{\theta / N}$
coupling g^2 fixed	${g'}^2 N = g^2$
$p_1 = \frac{n_1}{R}$ fixed, $e_2 = 0$	${p_1}'_{\mathrm{SU}(N)/Z_N} = p_1 = \frac{n'_1 - \frac{m}{N} e_2}{R'}$ $n'_1 = e'_2 = n_1 \sim \sqrt{N}$

Table 1: ???

the large-N limit, and equal to that of the open Wilson line in the non-commutative description. The SU(N) component of the Wilson loop carries a fractional momentum $p'_{\mathrm{SU}(N)} = (n'_1 - \frac{m}{N} e'_2)/R'$. This momentum is equal to that of the dual open Wilson line, $p_1 = n_1/R$. To summarize, the relation between non-commutative QED and a large-N limit of U(N) gauge theory is given by

4.2 Non-vanishing string tension

The energy of a photon propagating on the non-commutative plane should depend on the momentum, θ (and perhaps the polarization). For simplicity, we take the momentum to be in the 1 direction. In the U(N) description, we must compute the energy of a state created by a fundamental Wilson loop of non-trivial homology, with transverse momentum n'_1 and winding number e'_2 satisfying $n'_1 = e'_2 \sim \sqrt{N}$. The length of the Wilson loop, $2\pi e'_2 R'$ is fixed in the large-N limit, and is equal to the length of the open Wilson line in the non-commutative description. In order for this limit to be smooth, the QCD string tension should be fixed as $N \to \infty$ with $R' \sim 1/\sqrt{N}$. The dispersion relation depends crucially on whether this tension vanishes or not.

One possible signal of vanishing tension would be a phase transition in which the $Z_N \times Z_N$ symmetry of large gauge transformations on the commutative torus is broken below a critical radius. This would correspond to the condensation of electric flux, since the electric fluxes are charges associated to the $Z_N \times Z_N$ symmetry. Naively, one might have expected such a transition upon shrinking the radius of the torus, in analogy with a finite temperature deconfinement transition [22]. While we do indeed construct non-commutative QED as a small radius limit of pure QCD (at fixed Λ), this limit is accompanied by a large-N limit which invalidates the analogy.

To see that the usual arguments for symmetry breaking are no longer applicable, let us first recall the argument of [22], which applies to an SU(N)/Z_N theory in the absence of magnetic flux, and which involves shrinking the compactified euclidean time direction keeping N fixed. Note that when the compact direction is time, the interpretation of Z_N symmetry breaking is different, i.e. finite temperature deconfinement. When the compactified directions is spatial, Z_N symmetry breaking implies condensation of electric flux in the compactified direction. At high temperature, the modes on the S^1 may be taken to be constant. The action can then be written

$$S = \frac{1}{g^2} \int d^3x \, \mathrm{tr} \left[\frac{1}{\beta} D_i h D_i h^\dagger + \beta F_{ij}^2 \right] , \tag{4.2}$$

where $\beta = 2\pi R$ is the radius of the S^1, and h is the SU(N) valued holonomy in the time direction, $h = P\exp(i\oint A_0)$. When restricted to the zero mode, $h = \exp(i\beta A_0)$. The Z_N center symmetry acts as $h \to zh$ where $z \epsilon Z_N$. A gauge invariant order parameter is $\rho \equiv \mathrm{tr}\, h$. At small β, the path integral is dominated by configurations with $h^\dagger D_i h = 0$. This equation is satisfied for a generic A_i only if the holonomy h is a constant element in the center of SU(N). For sufficiently small β the effective potential for ρ has minima at traces of elements of Z_N, implying that the Z_N symmetry is spontaneously broken. As $\beta \to 0$ the vacuum is a superposition of of all N electric fluxes. We emphasize that this argument is a weak coupling argument, valid for $\beta\Lambda \ll 1$. The symmetry breaking transition occurs at $\beta\Lambda = c(N)$ where $c(N)$ is a coefficient that must be determined by non-perturbative means.

The above arguments do not extend to the case in which there is a non-zero magnetic flux, $m = N - 1$, and two S^1's are shrunk to zero with $R' \sim 1/\sqrt{N}$, since one can no longer truncate to the zero modes in the compact directions. Below the scale $1/R'$, the surviving degrees of freedom are wrapped Wilson loops with non-zero fractional momenta [20], quantized in units of $1/N$. Recall that the fractional component of the momentum is me'/N where e' is the winding number or electric flux. Furthermore the winding Wilson loops are dual to open Wilson lines with integer momenta on the non-commutative torus. This corresponds to the fact that the $Z_N \times Z_N$ large gauge transformations on the commutative torus are dual to to discrete translations on the non-commutative torus [11]. Thus $Z_N \times Z_N$ symmetry breaking would imply spontaneously broken translation invariance in the non-commutative description [23].[1]

To discover electric flux condensation by weak coupling methods requires that $\Lambda^2\theta \ll 1$. In this case the effective coupling is small in both the UV and IR. Naively one might also try to consider the $\Lambda^2\theta \gg 1$ region and set all the relevent scales (other than $1/\sqrt{\theta}$) large compared to Λ. However this is impossible when $\Lambda^2\theta \gg 1$ due to the relation between the length and the momentum of a Wilson loop or open Wilson line, $L = p\theta = me'/NR'$. The momentum and inverse length can not be simultaneously large compared to Λ. This is to be distinguished from the more familiar situation with no flux and fixed N, in which the relevent scale is just the length of the wrapped Wilson loop. In this case weak coupling arguments apply when $\Lambda R' \ll 1$. However in the case of interest to us, one can study the question of electric flux condensation perturbatively only when $\Lambda^2\theta \ll 1$. This was done in [23], where it was found that there are two competing phenomena. The non-zero momenta of wrapped Wilson loops (or open Wilson lines in the non-commutative description) gives a positive contribution to the electric flux energy which tends to prevent condensation. However this effect is overcome by terms related to UV/IR mixing in the non-commutative description, giving a tachyonic instability. This instability is indicative of electric flux condensation in the commutative description. The stable vacuum has not been studied, and it is possible that it does not have a smooth limit as $N \to \infty$, in which case pure QED on the non-commutative plane is ill defined for $\Lambda^2\theta \ll 1$.

[1]There are also examples of of spontaneously broken translation invariance in non-commutative scalar field theories [24, 25].

We do not currently have a good understanding of the dynamics in in the strong coupling region $\Lambda^2\theta \gg 1$. However we conjecture that the $Z_N \times Z_N$ symmetry is unbroken here. We will assume also that the string tension is non-zero and proportional to Λ^2. This assumption leads to rather interesting physical consequences. Note that at fixed N, an $R' \to 0$ limit would only give zero length wrapped strings, since electric fluxes annihilate modulo N. In the limit which we consider however, N is taken to infinity as $R' \sim 1/\sqrt{N} \to 0$, admitting wrapped stable strings of any length. We are interested in electric fluxes e'_i scaling like $\sqrt{N}$ (see table 1), since these are excitations with finite length and fractional momentum. In the non-commutative description, these correspond to open Wilson lines with the same momentum and length, only the momentum is no longer fractional by virtue of the relation $R = NR'$. The contribution of the string tension to the energy of such a state is $E \sim e_i \Lambda^2 R'$, which is finite as $N \to \infty$.

4.3 The dispersion relation

We seek the photon dispersion relation in the strongly coupled region $\Lambda^2\theta \gg 1$. Let us first consider an open Wilson line with momentum p_1 in the 1 direction of of the non-commutative torus, which is dual to a Wilson loop of the same length and momentum wrapping the 2 direction of the dual torus. Assuming a string tension $\sim \Lambda^2$, and that the length θp_1 of the open Wilson line is large compared to $1/\Lambda$, the dominant contribution to the energy $E \sim \Lambda^2 p_1 \theta$,

Let us also consider a photon propagating in the 3 direction transverse to the non-commutative plane. Modes which carry no momentum in the non-commutative directions are decoupled and free, as can be easily seen from the definition of the $*$-product (2.1). The dispersion relation for such a photon is $E = |p_3|$. Note that in the dual U(N) description, this corresponds to an excitation in the decoupled U(1) sector. The energy spectrum in this sector is given by

$$E = \sqrt{\left(\frac{k_1}{R'}\right)^2 + \left(\frac{k_2}{R'}\right)^2 + p_3^2}\,, \tag{4.3}$$

where k_1 and k_2 are integer momenta. In the small R' limit, the only U(1) modes which survive are those which have momentum only in the 3 direction.

The dispersion relation should be consistent with the above results, and with the violation of lorentz invariance by the rank two tensor $\theta^{\mu\nu}$. The simplest dispersion relation satisfying these requirements is

$$p^\mu p_\mu \sim \Lambda^4 p_\alpha \theta^{\alpha\beta}\theta_{\beta\gamma}p^\gamma\,. \tag{4.4}$$

A subtle point is that the definition of the four momentum p_μ for $\theta_{0i} \neq 0$ requires some care, since the usual hamiltonian formulation is not applicable. We interpret the expression (4.4) as giving a pole in a propagator computable in a path integral formalism. For non-zero $\theta_{12} = -\theta_{21} = \theta$ with all other components vanishing, the dispersion relation at strong coupling (4.4) is

$$p_0 = \sqrt{(p_1^2 + p_2^2)(\Lambda^4\theta^2) + p_3^2}\,. \tag{4.5}$$

In writing this expression, we have absorbed possible numerical factors into the definition of Λ. In the strong coupling regime in which the above expression is presumed valid ($\Lambda^2\theta \gg 1$ and $p\theta \gg 1/\Lambda$) light propagates "superluminally" in the non-commutative directions.

4.4 Including matter

We now consider the non-commutative U(1) gauge theory with charged matter. At energies below the mass of the lightest charge matter, the effective theory is the pure non-commutative U(1) theory discussed above, with a a dynamically generated scale Λ and a string tension $\sim \Lambda^2$. For sufficiently large momentum in the non-commutative plane, we expect the photon to become unstable and decay by production of charged pairs.

This instability becomes kinematically possible due to the modified dispersion relation (4.4). The argument giving the modified dispersion relation for the photon (4.4) is not applicable to charged matter, since the length of open Wilson lines bounded by charged sources is arbitrary and unrelated to the transverse momentum. We shall assume that unlike the photon, the electron has approximately the usual dispersion relation $p^2 = m_e^2$ at large momentum. Sufficiently long open Wilson lines, corresponding to high momentum photons, are then unstable against the production of electron-positron pairs via the process illustrated in figure 1. This is analogous to pair production in QCD as quark anti-quark pairs are pulled apart. Note that a similar instability occurs in conventional commutative QED when high energy photons propagate in a background magnetic field, although the mechanism is quite different.

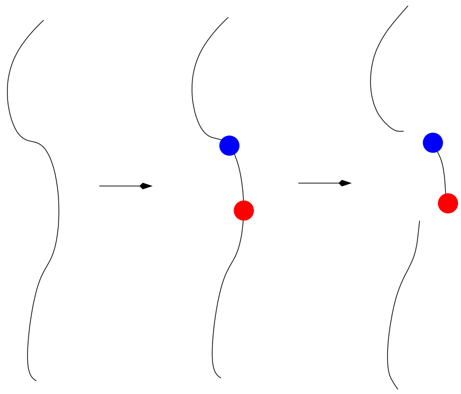

Figure 1: At large momentum, photons (open Wilson lines) in NC-QED with a single charged flavor are unstable. The process illustrated is $\gamma \to \gamma\gamma e_+ e_-$. This instability also exists for large momentum glueballs in NC-QCD, in which case the figure above corresponds to $G \to GG\pi$.

Given the dispersion relation (4.4), the stability bound on the momentum of a photon in non-commutative QED is given by

$$\Lambda^4 p_\alpha \theta^{\alpha\beta} \theta_{\beta\gamma} p^\gamma < 4m^2 \,, \tag{4.6}$$

where m is the mass of the lightest charged particle and Λ the dynamically generated scale of the effective theory at energies below m.

Analogous arguments also apply to the stability of glueballs in non-commutative QCD. Like photons in NC-QED, Glueballs in NC-QCD are created by open Wilson lines, with length proportional to the momentum in the non-commutative plane. In addition the usual instabilities, there is an enhanced instability beyond a critical momentum satisfying

$$\Lambda^4 p_\alpha \theta^{\alpha\beta} \theta_{\beta\gamma} p^\gamma = m_\pi^2 \,, \tag{4.7}$$

where m_π is the mass of the lightest pion and Λ^2 is the QCD string tension. Note that unlike NC-QED, the string tension remains non-zero even when the cutoff is small compared to the non-commutativity scale. However such a cutoff excludes momenta where this instability could be observed.

5. 't Hooft large-N limit vs. large-N–small-radius limit

The $N \to \infty$ limit of commutative U(N) gauge theory which is dual to the non-commutative U(1) theory differs from the conventional 't Hooft limit in the N dependent scaling of the compactification radius. As a consequence of this scaling, the $N \to \infty$ limit is not a planar limit for most observable quantities. Some exceptions are the beta function and anomalous dimensions, which are independent of the compactification in the commutative U(N) theory. This is consistent with what one expects for a U(1) theory on the non-commutative plane at finite θ, in which observables in general receive contributions from non-planar diagrams, while UV divergences arise only from planar diagrams. As we shall see later, A stringy limit, i.e. a truncation to the planar diagrams, arises in a certain $\theta \to \infty$ limit. This limit is T-dual to a conventional 't Hooft $N \to \infty$ limit with fixed m/N, where m is the magnetic flux on a torus of fixed size.

The usual correspondence between a $1/N$ expansion and a genus expansion arises when the lagrangian is of the form $\frac{N}{\lambda}\,\mathrm{tr}\, f(M)$, where the elements of the $N \times N$ matrix M are independent. Let us now consider the action of U(N) Yang Mills theory with a non-zero magnetic flux m satisfying $\gcd(N, m) = 1$. The gauge fields satisfy the boundary conditions (3.11). The general solution of (3.11) is

$$\tilde{A}^\mu_{ij} = \sum_{\mathbf{r}} a^\mu_{\mathbf{r}} (Q^{br_1} P^{r_2})_{ij} e^{i\pi \frac{b}{N} r_1 r_2} e^{i\mathbf{r}\cdot\mathbf{x}/(NR')} . \tag{5.1}$$

Written in this form, $\tilde{A}$ does not have N^2 independent matrix elements for a given momentum. There is an apparent reduction in the number of degrees of freedom from N^2 to 1. This can be understood in terms of the duality relating the theory to non-commutative QED. Nevertheless, for sufficiently large momentum (and short distances) the boundary conditions corresponding to the flux m should have no effect on the physics. This can be seen rather explicitly in the following way. Roughly speaking, one can exchange fractional momentum modes for matrix degrees of freedom. To do so, one writes $\mathbf{r} = N\mathbf{k} + \mathbf{l}$ with $\mathbf{l}$ defined modulo N, and defines a new "gauge field" $\mathcal{A}$ as follows;

$$\mathcal{A}^\mu_{ij} \equiv \sum_{\mathbf{k}} a^\mu_{\mathbf{k}N+\mathbf{l}} (Q^{bl_1} P^{l_2})_{ij} e^{i\pi \frac{b}{N} r_1 r_2} e^{i\mathbf{k}\cdot\mathbf{x}/(R')} . \tag{5.2}$$

Since the P and Q matrices are complete, one can write

$$\mathcal{A}^\mu_{ij} = \sum_{\mathbf{k}} f^\mu_{ij}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}/R'} \tag{5.3}$$

with no constraints on $f^\mu_{ij}(\mathbf{k})$. If one considers a Yang-Mills action with the gauge field $\mathcal{A}$, the usual 't Hooft large-N expansion applies. In momentum space, the difference between the Yang Mills action with the gauge field $\mathcal{A}$ or $\tilde{A}$ lies in terms proportional to $(k+j/N)/R'$ as opposed to k/R'. At very large k, i.e. $k \gg j/N$, there is no difference, as expected. However if one scales the radius of the torus as $R' \sim 1/\sqrt{N}$, then the modes that do not decouple as $N \to \infty$ have finite $(k+j/N)/R'$, which is impossible if $k \gg j/N$. Thus the action is not of the form which gives a genus expansion in $1/N$.

6. An open string theory from $\theta \to \infty$

It was suggested in [3] that the $\theta \to \infty$ limit of non-commutative field theory gives rise to a string theory. The reasons for this observation were as follows. The Feynman graphs of a non-commutative theory can be written in a double line notation, just as in a theory with matrix indices. One can then organize graphs according to their genus. In the large-θ limit the non-planar graphs are suppressed due to very oscillatory phases in the loop integrations. The genus expansion is presumably an l_s/θ expansion where l_s is an appropriate length scale.

For a gauge theory, the physical interpretation of a string theory arising in the $\theta \to \infty$ limit is not obvious for the following reasons. Gauge theory on the non-commutative plane at finite θ has open string-like objects, i.e. open Wilson lines. However one would naively not expect such objects to exist in the $\theta \to \infty$ limit, since their length in the non-commutative plane is $L = |\theta^{ij} p_j|$ which diverges as $\theta \to \infty$. The remaining gauge invariant operators are Wilson loops, i.e. closed strings, at zero momentum. Thus in the $\theta \to \infty$ limit, one might expect a closed string theory in which space-time (in the non-commutative directions) has vanished and only global operators survive. Elaborating on a suggestion appearing in [11], we will see below that there is a particular $\theta \to \infty$ limit of non-commutative gauge theory in which open strings do survive.

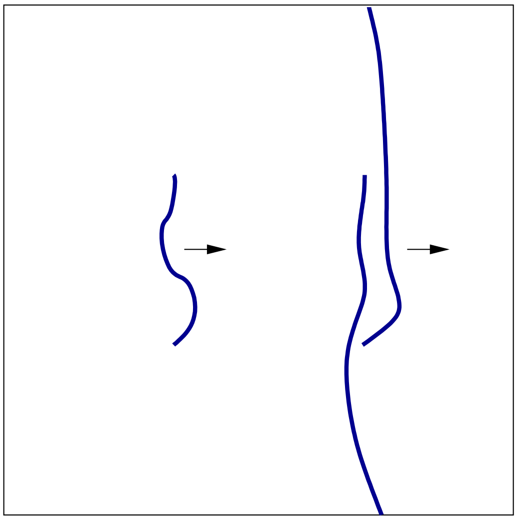

Figure 2: Two open Wilson lines with the same transverse momentum. Their length differs by $2\pi R n$ where n is the difference in electric fluxes.

One can obtain finite length open strings at infinite θ by compactifying the non-commutative plane on a torus and scaling θ and the radius to infinity simultaneously. Open Wilson lines on the torus are characterized by a winding number (or electric flux) in addition to their transverse momentum ([11]). This is illustrated in figure 2. Gauge invariance requires the length of an open Wilson line propagating in, say, the 1 direction of the non-commutative 12 torus to be $L = \theta^{21} p_1 + 2\pi R e_2$ where the integer e_2 is the electric flux. At finite p_1, the first term in this expression for the length is infinite as $\theta \to \infty$.

On the other hand, the second term is also infinite in the $R \to \infty$ if e_2 is non-zero. It is possible to arrange the limit in such a way that the sum of these two terms is finite.

One way take such a limit is to set $\theta = -\frac{c}{N} 2\pi R^2$ with $c = 1$ and take $N \to \infty$ with $R/N = R'$ fixed. The resulting theory is dual to the large-N limit of a $\mathrm{U}(N)$ Yang Mills theory with flux $m = N - 1$, and *fixed* radius R'. Note that this is now a conventional 't Hooft large-N limit, with magnetic flux such that m/N is fixed. Consider a holonomy in the $\mathrm{U}(N)$ theory which creates a state with electric flux e'_2 and transverse momentum n'_1, which are both held fixed in the large-N limit. Since m/N and R' are fixed in this limit, the length of the open string in the non-commutative description

$$L = \theta^{21} p_1 + 2\pi R e_2 = 2\pi R' e'_2 \tag{6.1}$$

is fixed, as is the transverse momentum

$$p_1 = \frac{n_1}{R} = \frac{n'_1 - \frac{m}{N} e'_2}{R'} \,. \tag{6.2}$$

where $m = N - 1$. Thus we obtain well defined open string states in the $\theta \sim N \to \infty$ limit with $R \sim N$.

Since we are dealing with a standard 't Hooft limit in the dual $\mathrm{U}(N)$ description, the genus expansion is an expansion in powers of $1/N$, which plays the role of the string coupling g_s. In the non-commutative description, one would expect the genus expansion to be an expansion in l_s^2/θ, where l_s is an appropriate length scale. Since $1/N = {R'}^2/\theta$, we conclude that $l_s = R'$. Note that the string scale l_s is related to the compactification scale R by $l_s = \theta/R$. To summarize,

$$\begin{aligned} l_s &= \frac{\theta}{R} \\ g_s &= \frac{l_s^2}{\theta} \,. \end{aligned} \tag{6.3}$$

The length of open strings is quantized in units of l_s, due to (6.1), The momentum is also constrained via (6.2) to be $p_1 = L_2/(2\pi\theta) + j/l_s$, where j is an integer.[2]

Strictly speaking, unless the open string theory is continuous in θ, our discussion is only applicable to the case in which $\Theta = 1/N$. Continuity in θ should not be taken for granted when the space is compact [4, 11], which is necessarily the case if $g_s \neq 0$ and l_s is finite.

In some instances, one can compute the correlator of the dual open string states exactly at $g_s = 0$ by using the $\mathrm{U}(\infty)$ dual. This was done in [11], and we repeat the result here, writing it in terms of non-commutative variables. For pure euclidean NC-QED in two dimensions, this correlator is defined by

$$G_2(L_2, p_1) \equiv \left\langle \left(\frac{1}{R} \int d^2x P^* e^{i \int_C A_2} * e^{i p_1 x^1} \right) \left(\frac{1}{R} \int d^2y P \left(e^{-i \int_C A_2} \right) * e^{-i p_1 y^1} \right) \right\rangle, \tag{6.4}$$

where C is the path from (x^1, x^2) to $(x^1, x^2 + L_2)$. At $g_s \to 0$ this can be computed in the

[2] To see this write, $n'_1 = e'_2 + j$ in (6.2).

$N \to \infty$ limit of the U(N) description, giving

$$G_2(L_2, p_1) = e_2' R' \int_{-\infty}^{\infty} dy e^{i(n_1' N - m e_2')y/(NR')} e^{-g'^2 N e_2' R' |y|} . \tag{6.5}$$

In terms of non-commutative variables, this may be rewritten as

$$G_2(L_2, p_1) = \frac{2g^2 L_2^2}{(g^2 L_2)^2 + p_1^2} , \tag{6.6}$$

where g is the gauge coupling of NC-QED. In the two dimensional case, it is possible to compute the $1/N$ corrections although we shall not do so here.

In the $\mathcal{N} = 4$ version of NC-QED in four dimensions, the equal time correlator of open strings in the non-commutative plane can be computed at strong coupling using the dual U(N) description and the AdS-CFT correspondence [11]. In terms of non-commutative variables, one obtains

$$G_2(L_2, p_1) = L_2 \int_{-\infty}^{+\infty} dy e^{ip_1 y} \exp\left(\frac{L_2 (4\pi)^2 (2g^2)^{1/2}}{\Gamma(\frac{1}{4})^4 |y|}\right) . \tag{6.7}$$

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